GLOBAL INSTABILITY IN SEPARATED INCOMPRESSIBLE LAMINAR BOUNDARY LAYER

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ABSTRACT

The purpose of this paper is to study the spatial structure of global instabilities, solutions of the partial derivative eigenvalue problem resulting from a nonparallel linear instability analysis of the incompressible Navier-Stokes equations. A family of Falkner-Skan profiles is analyzed for the onset of absolute instability as the magnitude of the reversed flow and this family of profiles was used to construct generic models of separation bubbles and study the possible onset of global instability. This study will tackle the existence of global instabilites in a boundary layer flow with a recirculating bubble.

INTRODUCTION

Conventional linear instability theory, herein referred to as Orr-Sommerfeld (OS) theory, can relate the essentially nonlinear three-dimensional phenomenon of turbulence to exponential amplification of small- amplitude perturbations at certain frequency and Reynolds number. The limitation of this analysis is that the steady laminar basic flow upon which disturbances are developing is taken to be an one-dimensional profile, with the flow being independant of the other two spatial directions which are treated as periodic. Such an assumption restricts the realm of applicability of the OS theory to a small class of steady laminar flows. If the condition of periodicity is one of the two spatial directions may be relaxed on physical grounds into one of slow growth, the concept of "Multiple Scales analysis" (MSA) may be applied. However, only a weak nonparallel flow can be studied with this approach. The restriction imposed on the basic flows by the assumptions inherent in the OS and MSA analyses may be relaxed by considering an extension of the classic linear theory in which the condition of spatial homogeneity in the laminar basic state is required of one spatial direction only, while the two other spatial directions are fully resolved. The linearised system of equations which results may be recast in the form of an eigenvalue problem in a maner formally analogous to that of the one-dimensional OS theory. This approach permits to tackle the case of strongly nonparallel flows and thus essentially concerns the study of two-dimensional instabilities. However, it is important to specify a significant difference between two concepts being able to be approached by this theory: the concept of two-dimensional instabilities and global instabilities. When the basic flow is naturally two-dimensional as, for example, in the case of a rectangular duct flow (Tatsumi & Yoshimura, 1990), the study of these instabilities is a "simple" extension of the Poiseuille case. In this case, the inhomogeneous space directions are transverse with the flow direction, one can speak then about two-dimensional instability. The stability analysis can be carried out by a temporal or spatial analysis, and thus we have a local convective two-dimensional instability. The case where the basic flow is strongly nonparallel in the streamwise flow as, for example, the beginning of the Taylor flow (Griffond, 2002) or the attachment line flow (Lin & Malik, 1996), the concept of local instability loses sense, in this case we can speak about two-dimensional or global instability in the fact where resulting instability is valid on the all field. However, there is a case where the concept of global instability takes a different significance. When the flow is slightly nonparallel (in this case the local instability concept keeps all its significance) for example in the separated boundary layer, Chomaz *et al.* (1991) have shown that intrinsic global instability could exist if the basic flow is absolutely unstable in a non zero restricted extension zone.

The principal objective of this analysis is initially to study the global instability of a laminar separated boundary layer solution to the Falkner-Skan equation. This analysis will show that although the flow is not absolutely unstable, it can nevertheless be globally unstable, showing of this fact a dissension with the Chomaz *et al.* (1991) results.

BASIC FLOW

The general equations of motion for the instantaneous flow are the Navier-Stokes equations. The present stability theory is based on the classical small perturbations technique where the instantaneous flow is the superposition of the known mean flow and unknown fluctuations. All the instantaneous physical quantities $\boldsymbol{Z}$ (velocity and pressure) are decomposed into a mean value and a fluctuating one:

$$\boldsymbol{Z}(x,y,z,t) = \overline{\boldsymbol{Z}}(x,y) + \boldsymbol{Z}_f(x,y,z,t), \quad (1)$$

where the mean flow $\overline{\boldsymbol{Z}}$ is not *a priori* supposed to be weakly parallel, i.e. it depends on the x and y direction.
We consider a two-dimensional incompressible boundary layer, with a non zero pressure gradient on finished portion $\mathcal{D}$ in x ($\mathcal{D} = [x_0; x_4]$). We assume that the mean flow is governed by the incompressible boundary layer equations. The x-axis is taken along the direction of the flow, the y-axis normal to the flow. Let ($\overline{U}$,$\overline{V}$,0) be the velocity components in (x, y, z) directions, respectively. From a certain x-coordinate, it is well-known that the incompressible boundary layer equations of plane plate, with or without pressure gradient can be reduce to Falkner-Skan similarity equations.

$$f''' + ff'' + \beta\left(1 - f'^2\right) = 0, \quad (2)$$

with the boundary conditions

$$\eta = 0,\ \ f = f' = 0 \ \text{ et } \ \eta \to \infty \ \ f' \to 1. \quad (3)$$

where $(.)^{'} = d(.)/d\eta$ and $f^{'}(\eta) = \overline{U}/\overline{U}_e(\eta)$ with $\eta = y/\Delta(x)$. β represents the reduced pressure gradient and it is related with the physical pressure gradient by $\beta/(2-\beta) = -x/(\rho_e \overline{U}_e^2) d\overline{p}/dx$. Such solutions exist if the external velocity $\overline{U}_e(x)$ follows the law $\overline{U}_e(x) = kx^{\beta/(2-\beta)}$, k represents the external gradient velocity at the origin. $\Delta(x)$ is given by $\Delta(x) = x\sqrt{2-\beta}/\sqrt{R_{e_x}}$ with $R_{e_x} = \overline{\rho}_e \overline{U}_e x/\overline{\mu}_e$. The transversal velocity is given by:

$$\frac{\overline{V}(\eta)}{\overline{U}_e} = \sqrt{\frac{1}{2-\beta}\frac{1}{R_{e_x}}}\left(\frac{1-\beta}{2-\beta}\eta f^{'} - \frac{1}{2-\beta}f\right)$$

The equation (2) is solved by using a shooting method on the value of the skin friction coefficient in order to obtain the attached and separated solution ($\beta < 0$, $f^{''}(0) < 0$). Fig. 1 shows the evolution of β according to x on the curve giving the evolution of the skin friction coefficient versus to β. Fig. 2 shows a sketch presenting the boundary layer and

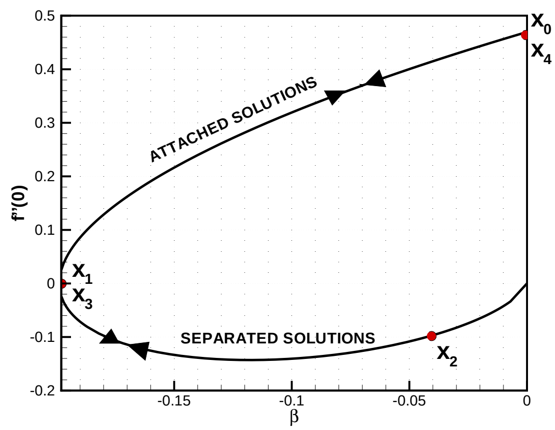

Figure 1: Evolution of skin friction coefficient $f^{''}(0)$ versus β parametred by the x-abscissa on the plate plane.

the different notations used. In particular, the different x-coordinates x_j $j = 0, ..., 4$ used in the calculation of stability and the definition of the reduced pression gradient are defined. In order to obtain a boundary layer bubble on a plane

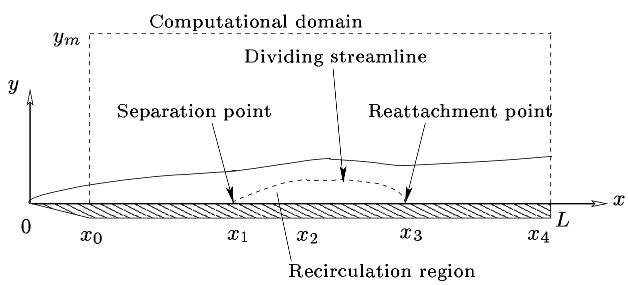

Figure 2: shows a sketch presenting the boundary layer and the different notations used.

plate, the reduced pressure gradient $\beta(x)$ is a function to x checking: $\beta(x) = \sum_{i=1}^{10} b_i x^i$ where the b_i are determined by the following relations $\beta(x_0) = \beta(x_4) = 0$, $\beta(x_1) = \beta(x_3) = -0.1988$, $\beta(x_2) = -0.04$ and $\partial\beta(x_i)/\partial x = 0$, $i = 0, \ldots, 4$. The choice of different x-coordinate is arbitrary, the plane plate length is chosen such as $x_4 = L$, $x_0 = 0$, $x_1 = L/3$, $x_2 = L/2$, $x_3 = 2L/3$ and $R_{e_L} = 10^5$. The choice of $\beta(x_2) = -0.04$ correspond to the maximum of the backflow velocity.

THEORETICAL APPROACH

Perturbation form

The inherent restriction of slow growth of the basic flow in the streamwise direction x assumed in the multiple scales approach may be relaxed by fully resolving both this as well as the wall-normal, y, spatial directions. A solution to the equations of motion $\boldsymbol{Z}(x, y, z, t)$ is decomposed into a steady two-dimensional basic and an unsteady three-dimensional perturbation component according to (1). According to the homogeneous form of the basic flow in cross-flow direction z, the perturbation can be described as a normal mode:

$$\boldsymbol{Z}_f(x, y, z, t) = \varepsilon \check{\boldsymbol{Z}}(x, y) e^{i(kz - \omega t)} + c.c., \ \varepsilon \ll 1, \quad (4)$$

where $\check{\boldsymbol{Z}}(x, y) = (\check{u}, \check{v}, \check{w}, \check{p})^t$ is the eigenfunction vector.

Stability equations

The decomposition is substituted into the incompressible equations of motion. In the temporal framework, as considered here, k is a real wavenumber parameter and ω is the seeked complex eigenvalue whose real part indicates frequency and whose positive imaginary part is the growth rate in time t. The imaginary unit is $i = \sqrt{-1}$ and $c.c.$ denotes complex conjugate in order for the physical field to remain real. Linearisation about $\overline{\boldsymbol{Z}}$ follows, based on the argument of smallness of ε and the basic flow terms, themselves satisfying the equations of motion, are subtracted out. With the perturbation (4), the linearised Navier-Stokes equations become a partial differential system:

$$\left[\boldsymbol{M}_1 \frac{\partial^2}{\partial x^2} + \boldsymbol{M}_2 \frac{\partial^2}{\partial y^2} + \boldsymbol{M}_4 \frac{\partial}{\partial x} + \boldsymbol{M}_5 \frac{\partial}{\partial y} + \boldsymbol{M}_6\right] \check{\boldsymbol{Z}} = 0, \quad (5)$$

where $\boldsymbol{M}_j$, $j = 1, \ldots, 6$ are real or complex (4×4) matrices. The explicit form of the different matrices $\boldsymbol{M}_j$ are deferred in appendix .

Stability equations and the basic-flow are nondimensionalized by global characteristic scales like upstream freestream velocity $U_e(x_0)$ and by a reference length L. $R_e = U_e(x_0)L/\nu_e$ is the global Reynolds number.

Boundary conditions

A study of the mathematical nature of the partial-derivative equations system shows that it is elliptic system. Moreover, in the case of an open flow system, the issue of boundary conditions must be considered carefully with respect to the plausible physical situations that one wished to model.

In the y direction: on the wall and at the far-field, the usual viscous conditions is employed

$$[\check{u}, \check{v}, \check{w}](x, 0) = 0 \text{ and } \frac{\partial \check{p}}{\partial y}(x, 0) = 0, \ \forall x \in [x_0, x_4]$$
$$\lim_{y \to y_m} [\check{u}, \check{v}, \check{w}](x, y) = 0 \text{ and } \lim_{y \to y_m} \frac{\partial \check{p}}{\partial y}(x, y) = 0, \ \forall x \in [x_0, x_4]. \quad (6)$$

In the x direction: Theoretically the different boundary layer flows that we will study in this paper are convectively unstable at inflow and outflow to the computation domain. Consequently, the eigenfunction vector which defines the streamwise structure of the disturbance field associated with

a global instability satisfies homogeneous boundary conditions at inflow and at outflow like the vanishing of the disturbance field asymptotically provided the flow is stable at $x = x_0$ and $x = x_4$.
In this paper, we will be interested only in global instabilities and not in local instabilities like the Tollmien-Schlichting waves present obviously from the beginning to the boundary layer. These instabilities are not computed in a present approach and we will thus not take account of their influences on the nature of the boundary layer, in particular in the separated zone where it is well-known that the existence of the inflection point in the velocity profile is the high instability sources likely to start the transition. We have adressed this problem by imposing homogeneous Dirichlet conditions on all perturbations at inflow, thus prohibiting any disturbances to enter the calculation domain and allowing only those generated by the presence of the steady laminar separation bubble.

Finally, at the outflow boundary linear extrapolation of all disturbance quantities from interior of the integration domain is considered.

$$[\breve{u},\breve{v},\breve{w}](0,y)=0 \text{ and } \breve{p}(0,y)=0,\ \forall y\in[0,y_m]$$
$$[\breve{u},\breve{v},\breve{w}](x_4,y) \text{ are extrapolated and } \breve{p}(x_4,y) \text{ verify :}$$
$$\frac{\partial\breve{p}}{\partial x}=\frac{1}{R_e}\frac{\partial^2\breve{u}}{\partial x^2}+\frac{1}{R_e}\frac{\partial^2\breve{v}}{\partial y^2}-\overline{U}\frac{\partial\breve{u}}{\partial x}-\overline{V}\frac{\partial\breve{v}}{\partial y}-\frac{\partial\overline{V}}{\partial y}+ik\overline{U}\breve{w}-\left(\frac{\partial\overline{U}}{\partial x}+\frac{k^2}{R_e}-i\omega\right)\breve{u},\ \forall y\in[0,y_m]. \quad (7)$$

NUMERICAL APPROACH

The partial differential stability equations (5) are discretised to use an algorithm based on the collocation method by n_x and n_y order Chesbyshev polynomials respectively. For more details on the method see Malik (1990). Finally, Eqs (5) with the boundary conditions may be represented as

$$(\boldsymbol{A}-\omega\boldsymbol{B})\,\boldsymbol{\phi}=0, \quad (8)$$

with $\boldsymbol{\phi} = \left\{\{\breve{\boldsymbol{Z}}_{ij}\}_{j=0,\cdots,n_y}\right\}_{i=0,\cdots,n_x}$ The algebraic system (8) represents the discretized eigenvalue problem. The Chebyshev interval $(\zeta_i,\xi_j)\in[0,1]\times[0,1]$ is transformed to the computational domain $\mathcal{D}$ by use of the mapping in x and y directions respectively

$$x=\frac{1}{\pi}\arccos(\zeta)\ \text{(uniform grid)}$$
$$y=a\frac{\xi+1}{b-\xi},\ \text{with } a=\frac{y_a y_m}{(y_m-2y_a)} \text{ and } b=1+\frac{2a}{y_m}. \quad (9)$$

In your case $y_a \simeq \delta$, where δ is the boundary layer thickness and $y_m \simeq 30\delta$. A standard eigenvalue subroutine may now be used to compute the $\dim(\boldsymbol{Z})\times(n_x+1)\times(n_y+1)$ eigenvalues. Two methods were used to solve the equation (8): a local method based on a shooting method with a classical Newton-Raphson algorithm and a global method, where the discretized operator spectrum is computed by an Arnoldi algorithm.

RESULTS

Quasi-analytical global instabilities

Absolute instabilities. As envisaged by the previous studies (Hammond & Redekopp, (1998), Robinet, (2001)), the bubble flow solution to the Falkner-Skan equation is not absolutely unstable for all frequency and Reynolds number. The figure 3 gives the evolution of the absolute circular frequency ($\mathrm{Re}(\omega_0)$) and the absolute temporal amplification rate ($\mathrm{Im}(\omega_0)$) according to X. For all X, the amplification rate is always negative and consequently the basic flow is not absolutely unstable.

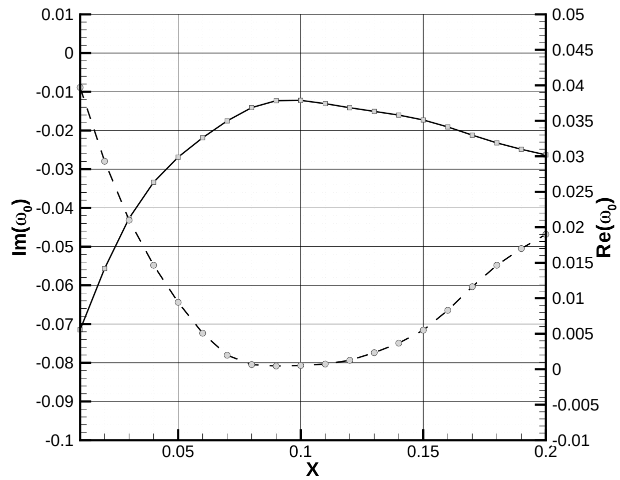

Figure 3: Absolute growth rate $\mathrm{Im}(\omega_0)$ and absolute frequency versus X, Solide line: $\mathrm{Im}(\omega)$ and dashed line: $\mathrm{Re}(\omega)$.

Global instabilities. The global stability analysis of laminar bubble can be studied by another approach where the weakly nonparallel flow ($y=\mathcal{O}(1)$, $x=\mathcal{O}(\varepsilon)$) is taken into account. This appraoch was already realized by Hammond & Redekopp (1998) for a symetric bubble, less representative of a real laminar bubble. This analysis consists in studying, at first, the convective-absolute transition from the local instability of the velocity profil $\overline{U}(y;X)$ where $X=x/\varepsilon$ with $\varepsilon \ll 1$ is taken as a parameter. Although our basic-flow is identical in these theoretical principles to that of the Hammond & Redekopp (1998), our laminar bubble is not symmetric as can see it the evolution of the different integral quantities (δ_1, θ and H) in figure 4. After to determine the absolute circular frequency $\omega_0(X)$ and the absolute wave number $\alpha_0(X)$ respectively, a linear evolution model for the streamwise structure of a possible global instability is developed and based on the following truncated approximation of the dispersion relation close to the absolute circular frequency

$$\omega(X)=\omega_0(X)+\frac{1}{2}\frac{\partial^2\omega}{\partial\alpha^2}(\alpha-\alpha_0(X))^2,\ \text{with } \frac{\partial\omega}{\partial\alpha}(\alpha_0)=0.$$

In the following, we suppose that $\partial^2\omega/\partial\alpha^2<0,\ \forall x\in\mathcal{D}$ to satisfy the causality. Contrary to the previous section, the transverse dependency of the fluctuation is not directly taken into account, it involves very indirectly through the values of $\omega_0(X)$ and $\alpha_0(X)$. Consistent with WKBJ theory, we assume that the parameters vary in the streamwise direction on the slow scale $X=\varepsilon x$. The linear evolution equation compatible with the dispersion relation can then

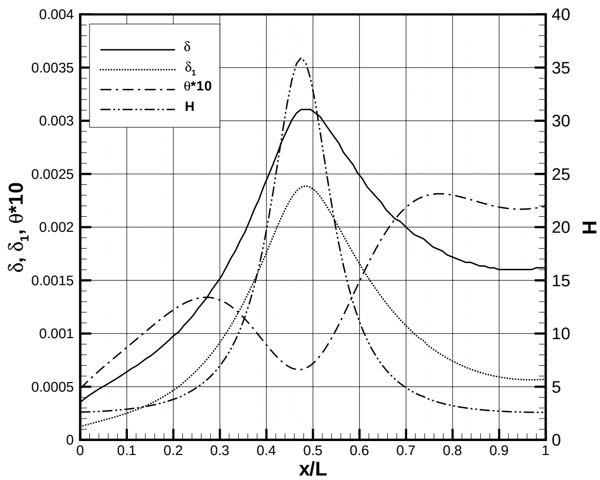

Figure 4: Evolution of Boundary layer thickness δ, displacement thickness δ_1, momentum thickness θ and shape factor H.

be expressed in the form

$$i\left(\frac{\partial A}{\partial t}-\alpha_0(X)\frac{\partial^2\omega}{\partial\alpha^2}\frac{\partial A}{\partial x}\right)+\frac{1}{2}\frac{\partial^2\omega}{\partial\alpha^2}\frac{\partial^2 A}{\partial x^2} - \left(\omega_0(X)+\frac{1}{2}\alpha_0^2(X)\frac{\partial^2\omega}{\partial\alpha^2}\right)A=0. \quad (10)$$

A global mode, solution of (10) can be search as

$$A(x,t;\varepsilon)=\phi(X;\varepsilon)e^{\frac{i}{\varepsilon}\int_{X_s}^{X}\alpha_0(X')dX'}e^{-i\omega_g t}, \quad (11)$$

where ω_g is the global mode frequency. The modal eigenfunction $\phi(X;\varepsilon)$ defines the streamwise structure of the fluctuating field associated with the global instability. Chomaz *et al.*(1991) employed the linear, variable-coefficient, Ginzburg-Landau equation (10) to determine generic criteria for onset of a global instability and its frequency selection. In particular, they demontrated that the saddle point position defined by $\partial\omega_0(X_s)/\partial X=0$, where X_s is an generalized abscissa (in general complex), is the appropriate position about which the variable coefficients in (10) should be referenced. They also showed that the leading-order estimate for the frequency of oscillation of the gravest global mode is given by $\mathrm{Re}(\omega_0(X_s))$. This eigenfunction must satisfy homogeneous boundary conditions, like the vanishing of the disturbance field in $x=x_0$ and $x=x_4$ because the flow is globally stable at these abscissa. Under these hypotheses, the evolution equation in X can be written as

$$\varepsilon^2\frac{d^2\phi}{dX^2}+\left[\frac{2(\omega_g-\omega_0(X))}{\partial^2\omega/\partial\alpha^2(X)}+i\varepsilon\frac{d\alpha_0}{dX}(X)\right]\phi(X;\varepsilon)=0, \quad \phi(X_0;\varepsilon)=\phi(X_4;\varepsilon)=0. \quad (12)$$

The functions $\omega_0(X)$, $\alpha_0(X)$ and $\partial^2\omega/\partial\alpha^2(X)$ are numerically determined and ε which represents the global estimate of the nonparallelism of the flow and can be estimated by $\epsilon=\max_{X\in[X_0,X_4]}\left(\max_{y\in[0,+\infty[}\frac{\overline{V}(X,y)}{\overline{U}(X,y)}\right)$, and must be $\varepsilon\ll 1$ so that the WKBJ approximation is valid. For this bubble, ε is in the most equal to 0.021. The differential equation (12) is solve with a classical spectral collocation method and the eigenvalues spectrum obtained is presented in Fig. 5. This analysis confirms the results obtained by the local instability studies, so that a global instability exists, it is necessary that there is a finished extension zone where the instability is absolute. However, in this reduced model, we focus exclusively on the streamwise development of the disturbance field, completely suppressing any dependence on the cross-stream (y) or eigenfunction direction.

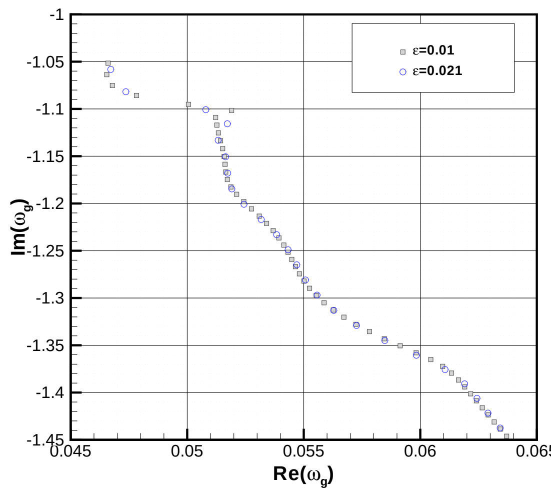

Figure 5: Discretize eigenvalues spectrum for the reduced model.

BiGlobal instabilities

In order to be independent of the different assumptions related on the linearized Ginzburg-Landau model, in particular from the absence of cross-stream dependence of the fluctuation, a two-dimensional stability analysis (5, 6, 7) is performed. These equations define an partial differential eigenvalue problem and yields the dispersion relation $\mathcal{F}(\omega,k;R_e)=0$. The following calculations were carried out for $n_x=50$, $n_y=46$ points. Figure 6 shows the eigenvalue spectrum in the neighbourhood of $\omega=0$ and for different values of spanwise wavenumbers k. Several unstable modes are observed, the stationary (Re(ω)=0) modes and the travelling modes (Re(ω) $\neq 0$). The travelling modes appear in symmetric pairs, indicating that there is no preferential direction in z. However, the most unstable mode is a three-dimensional stationary disturbance. Figure 7 presents the evolution versus spanwise wavenumber k of the global temporal amplification rate for the two most unstable modes (one stationary mode and one travelling mode). These unstable modes are not predicted by Chomaz approach. This difference can come from the taking into account of the cross-stream dependence at the same time as the streamwise dependence. Most of the activity in all disturbance eigenfunctions is confined within the boundary layer and is located in the separated zone. Figures 8 and 9 show the amplitude of the velocity components and the pressure distribution of the unstable stationary global mode. The three-dimensional character (where $|\tilde{w}|\neq 0$) is high around the reattachment point. The essential of the fluctuating activity ($|\tilde{u}|$, $|\tilde{v}|$) is confined in the center of the bubble. For the pressure fluctuation. The maximum in amplitude is located close to the shear layer and extend slightly outside the boundary layer. The effect of the presence of the

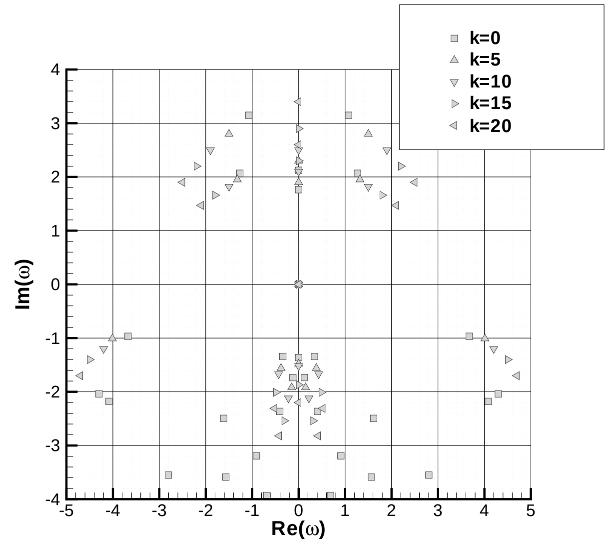

Figure 6: Discretized eigenvalues spectra for the separated boundary layer flow for different streamwise wavenumbers $k = 0, 5, 10, 15, 20$.

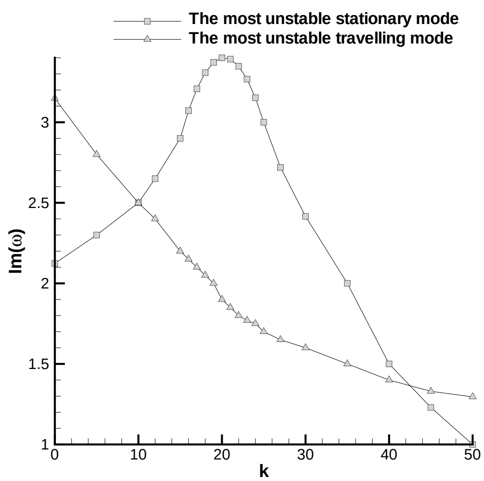

Figure 7: Evolution according to k of the amplification rate for the two most unstable modes.

stationary unstable mode on the laminar basic flow is very different for respectively the center of the bubble or the separation and reattachment points. While the separation and reattachment lines remain unaffected in the presence of a linearly unstable mode, the bubble center of the basic flow at sufficiently high amplitudes of the linearly unstable mode is divided into three distinct zones (separated, reattached and separated) where speeds are opposite signs. Between the two primary separation and reattachment point, a secondary separation and reattachment points are linearly generated. Fig. 10 shows this characteritics. Similar results have been obtained by Theofilis *et al.* (2000) for a laminar separated boundary layer resulting from a Navier-Stokes calculation. They have shown that there is a three-dimensional stationary global mode solution. However, these authors do not indicate if an absolute instability zone exists and they do not specify the link between this global instability and global instability concept defined by Chomaz (1991).

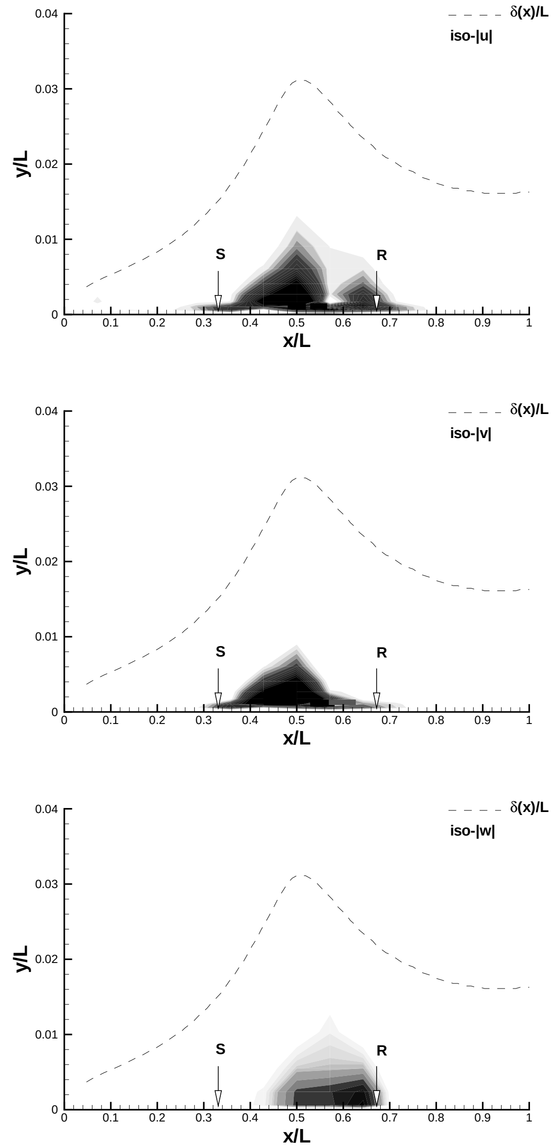

Figure 8: Normalised disturbance velocity componants amplitude of the unstable stationary global mode for $R_{e_L} = 10^5$ and $k = 20$.

CONCLUSIONS

The topic of this paper relates the existence of global instabilities, inaccessible to local analysis, in a separated flow. Solutions to the partial derivative eigenvalue problem have been performed, which correspond both to stationary and travelling global instabilities which are periodic in the spanwise z direction. These unstable global modes are not obtained by an quasi-analytical approach similar to that carried out by Hammond & Redekopp (1998). This difference seems to come from the taking into account the exact non-parallelism of the flow and permitting to treat the behavior of low frequency instabilities correctly and moreover the taking into account of the cross-stream dependance at the same time as the streamwise dependance should play an importante role in this discrepency between the BiGlobal approach and the Quasi-analytical global approach. From the point of view of control the laminar-turbulent transition by the inhibition of the instability mechanisms, for the separated

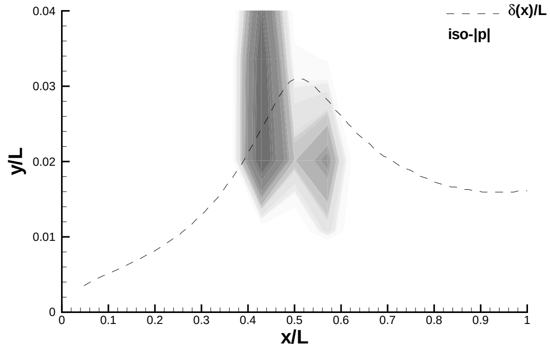

Figure 9: Normalised pressure distribution amplitude of the unstable stationary global mode for $R_{e_L} = 10^5$ and $k = 20$ (continued).

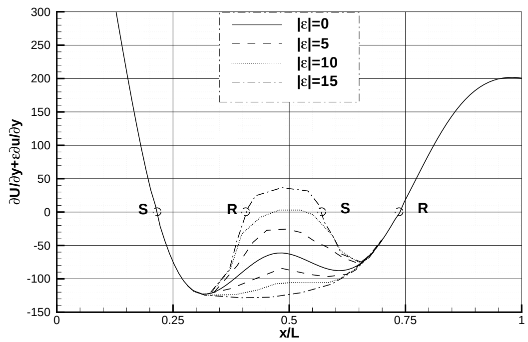

Figure 10: Wall-shear distribution due to the basic and different linear amounts of the disturbance streamwise velocity component.

flow, it is clear that current efforts focussing exclusively on convectives instabilities are bound to fail if the control mechanisms do not take account of the unstable global modes. More precisely, when the most unstable mode is stationary, the control methods based on the unstable mode frequencies are not adequate.

ACKNOWLEDGEMENT

Computing time was provided by Institut du Développement et des Ressources en Informatique Scientifique (DRIS)-CNRS.

APPENDIX: MATRIX FORMS

$$\boldsymbol{M}_{1,2} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \dfrac{1}{R_e} & 0 & 0 & 0 \\ 0 & \dfrac{1}{R_e} & 0 & 0 \\ 0 & 0 & \dfrac{1}{R_e} & 0 \end{pmatrix} \qquad \boldsymbol{M_3} \equiv \boldsymbol{0}$$

$$\boldsymbol{M}_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -\overline{U} & 0 & 0 & -1 \\ 0 & -\overline{U} & 0 & 0 \\ 0 & 0 & -\overline{U} & 0 \end{pmatrix}$$

$$\boldsymbol{M}_5 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -\overline{V} & 0 & 0 & 0 \\ 0 & -\overline{V} & 0 & -1 \\ 0 & 0 & -\overline{V} & 0 \end{pmatrix}$$

$$\boldsymbol{M}_6 = \begin{pmatrix} 0 & 0 & ik & 0 \\ \Lambda - \dfrac{\partial \overline{U}}{\partial x} & -\dfrac{\partial \overline{U}}{\partial y} & 0 & 0 \\ -\dfrac{\partial \overline{V}}{\partial x} & \Lambda - \dfrac{\partial \overline{V}}{\partial y} & 0 & 0 \\ -\dfrac{\partial \overline{W}}{\partial x} & -\dfrac{\partial \overline{W}}{\partial y} & \Lambda & -ik \end{pmatrix}$$

with $\Lambda = -\frac{k^2}{R_e} - i(k\overline{W} - \omega)$

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