TWO-DIMENSIONAL LOCAL INSTABILITY: COMPLETE EIGENVALUE SPECTRUM

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Abstract The study of "simple" flows such as Poiseuille flow have been for a long time studied and their simplicity has permit to highlight different instability mechanisms. More recently, several authors have shown, through transient growth studies and by the identification of the nonnormal character of the operator of Orr-Sommerfeld, that these two flows could have a subcritical dynamics. However, the assumption of one-dimensionality of the basic flow limits the comparison with the experiment where the basic flow is not exactly one-dimensional. Although different studies have been realized on the stability of a two-dimensional basic flow, all were interested only in the most unstable mode. In this present paper, the stability of the laminar flow in a rectangular duct of an arbitrary aspect ratio is investigated numerically with for objective the computation of the complete spectrum. This study will highlight strong similarities between the 1D and 2D spectra but, in spite of a powerful numerical method, will show the numerical limitations observed to obtain a spectrum converged in a sufficiently large field in ω.

Keywords: Two-dimensional instability, collocation spectral, eigenvalue problem, Poiseuille flow.

Introduction

Since many years it is well-known that for some flows the prediction of linear eigenvalue analysis fail to match most experiments. Poiseuille flow is this kind of flows. Recently it has emerged that the failure of eigenvalue analysis may more justly be attributable to the non-normality of the linearized system. It is a fact of linear algebra that even if all eigenvalues of a linear system are distinct and stable, inputs to that system may be amplifed by arbitrarily large factors if the eigenfunctions are not orthogonal.

S. C. Reddy and Henningson, 1993, have discovered that the operator that rise Poiseuille flow is in sense exponentially far from normal and Butler and Farrell, 1992, have shown that small perturbations to these flows may be amplified by factors of many thousands, even if all the eigenvalues are stable.

However, although these new approaches have considerably evolved the comprehension of the mechanisms of transition to the turbulence in open flows such as Poiseuille flow, the modeling of these flows remains rather simple. Indeed, for all of these flows only one spatial direction is supposed to be homogeneous, at least in an approximate sense. This restriction may be relaxed by considering an extension of the classic linear theory in which the condition of spatial homogeneity in the laminar basic state is required of one spatial direction only, while the other two spatial direction are resolved. This extension was already used by some authors like Tatsumi and Yoshimura, 1990, and Theofilis, 1998, Theofilis, 2000, for the calculation of instabilities in a rectangular duct flow and Theofilis et al., 2000, Theofilis, 2003, for many academic flows. However, all these studies related to the calculation of the most unstable modes and not of the complete spectrum of the discrete eigenvalues necessary for the computation of transient growth.
The objective of this paper is to study the structure of the discretized operator spectrum, to compare it with the spectrum when $A \rightarrow \infty$ and to evaluate the capacity of the computational techniques usually used to calculate the "whole" of the discrete eigenvalues spectrum in simple flow in a duct of rectangular cross-section: two-dimensional Poiseuille flow.

1. Decomposition of the Instantaneous Flow Field

The main idea of the linear stability theory is to split the flow field in two parts. The first part is of order O(1) and is called the basic flow (noted $\overline{\bullet}$). It is supposed to be steady and solution of the Navier Stokes equations. The second part is the disturbance term and is assumed to be of small amplitude. Considering that the basic flow is two-dimensional in (x, y) and the third direction is assumed to be homogeneous. In cartesian coordinates, each physical quantity $\boldsymbol{Z}$ of the flow field (velocity components, pressure, temperature, density) is thus written :

$$\boldsymbol{Z}(x, y, z, t) = \overline{\boldsymbol{Z}}(x, y) + \varepsilon \tilde{\boldsymbol{Z}}(x, y) e^{i(\beta z - \omega t)} + cc., \ \ \varepsilon \ll 1. \tag{1}$$

The aim is to determine the disturbance field. Filling the Navier Stokes equations with this ansatz, a new system is obtained, from which the basic flow terms can be dropped, as they already solve the equations. By linearising, the quadratic flow terms also disappear.

In the temporal framework, as considered here, β is a real wavenumber parameter and ω is the seeked complex eigenvalue whose real part indicates frequency and whose positive imaginary part is the growth rate in time t. The imaginary unit is $i = \sqrt{-1}$ and $c.c.$ denotes complex conjugate in order for the physical field to remain real. The solutions of the problem are thus seeked under the form of temporal waves behaving periodically in the z-direction, and of amplitude the unknown eigenfunction vector $\tilde{\boldsymbol{Z}}(x, y)$.

2. Basic Flows

We considers a duct with rectangular section $[-A, A] \times [-1, 1]$ following x and y directions respectively and of infinite extension in z direction. According to the geometry of the problem, one can suppose that the velocity field does not depend on z, the velocity is thus: $\overline{\mathbf{V}} = [0, 0, \overline{W}(x, y)]^t$ and pressure can be written as $\overline{P}(x, y, z)$. Under these assumption, the Navier Stokes equations can be written as Poisson equation $\partial^2\overline{W}/\partial x^2 + \partial^2\overline{W}/\partial y^2 = \lambda R_e$, and the pressure $\overline{P}$ is linear in the z-direction. where $\lambda = \partial\overline{P}/\partial z$ is a constant. This equation is discretized by spectral collocation méthod and solved by classical numerical method. The boundary conditions are for the two-dimensional Poiseuille flow: $\bar{W}(\pm A, y) = \bar{W}(x, \pm 1) = 0,\ \forall(x, y) \in [-A, A] \times [-1, 1]$.

3. Linearized Navier-Stokes (LNS) formulation

The decomposition (1) is substituted into the incompressible equations of motion. Linearisation about $\overline{\boldsymbol{Z}}$ follows, based on the argument of smallness of ε and the basic flow terms, themselves satisfying the equations of motion, are subtracted out. The linearised Navier-Stokes equations become a partial differential system and can be written in matricial form:

$$\left[\boldsymbol{M}_1\frac{\partial^2}{\partial x^2} + \boldsymbol{M}_2\frac{\partial^2}{\partial y^2} + \boldsymbol{M}_3\frac{\partial^2}{\partial x\partial y} + \boldsymbol{M}_4\frac{\partial}{\partial x} + \boldsymbol{M}_5\frac{\partial}{\partial y} + \boldsymbol{M}_6\right]\tilde{\boldsymbol{Z}} = 0, \quad (2)$$

with $\tilde{\boldsymbol{Z}}(x, y) = [\tilde{u}, \tilde{v}, \tilde{w}, \tilde{p}]^t\,(x, y)$ and where $\boldsymbol{M}_j$, $j = 1, \ldots, 6$ are real or complex (4×4) matrices. For more details, see Theofilis, 2003. In this formulation the boundary conditions are :

$$\begin{aligned} \tilde{u}(x, \pm 1) = \tilde{v}(x, \pm 1) = \tilde{w}(x, \pm 1) = 0 \text{ and } \frac{\partial\tilde{p}}{\partial y}(x, \pm 1) = 0,\ \forall x, \\ \tilde{u}(\pm A, y) = \tilde{v}(\pm A, y) = \tilde{w}(\pm A, y) = 0 \text{ and } \frac{\partial\tilde{p}}{\partial x}(\pm A, y) = 0,\ \forall y. \end{aligned} \quad (3)$$

4. Two-dimensional Orr-Sommerfeld (2DOS) formulation

Here we have a system of four variables. In order to spare computer memory, it is possible to combine the equations and obtain two two-variables systems: the two-dimensionnal extension of the Orr-Sommerfeld equation. Finally the EDP system can be written in the following matrix form:

$$\begin{aligned} &\left[\boldsymbol{M_1}\frac{\partial^4}{\partial x^4} + \boldsymbol{M_2}\frac{\partial^4}{\partial y^4} + \boldsymbol{M_3}\frac{\partial^4}{\partial x^3\partial y} + \boldsymbol{M_4}\frac{\partial^4}{\partial x^2\partial y^2} + \boldsymbol{M_5}\frac{\partial^4}{\partial x\partial y^3} + \right. \\ &\left.\boldsymbol{M_6}\frac{\partial^2}{\partial x^2} + \boldsymbol{M_7}\frac{\partial^2}{\partial y^2} + \boldsymbol{M_8}\frac{\partial^2}{\partial x\partial y} + \boldsymbol{M_9}\frac{\partial}{\partial x} + \boldsymbol{M_{10}}\frac{\partial}{\partial y} + \boldsymbol{M_{11}}\right]\tilde{\boldsymbol{Z}} = 0, \end{aligned} \quad (4)$$

with $\tilde{\boldsymbol{Z}}(x,y) = [\tilde{u}(x,y), \tilde{v}(x,y)]^t$. For more details, see Tatsumi and Yoshimura, 1990. In this formulation the boundary conditions are :

$$\begin{cases} \forall x,\ \tilde{u}(x,\pm 1) = \tilde{v}(x,\pm 1) = \dfrac{\partial \tilde{v}}{\partial y}(x,\pm 1) = 0, \\ \forall y,\ \tilde{u}(\pm A,y) = \tilde{v}(\pm A,y) = \dfrac{\partial \tilde{u}}{\partial x}(\pm A,y) = 0. \end{cases} \tag{5}$$

The problem so formulated is an eigenvalue problem. In other words, it exist an implicit dispersion relation where the different parameters are connected between them. The resolution of the eigenvalue problem brings the family of eigenvalues ω_j and the corresponding eigenfunctions $\tilde{\boldsymbol{Z}}$. The real part $\Re(\omega_j)$ is the disturbance wave frequency and its imaginary part $\Im(\omega_j)$ is its growth or damping rate.

5. Numerical Methods

Spectral Chebychev collocation method is used in both direction. The grid is built with Gauss-Lobatto nodes. The principal numerical difficulties lie in the resolution of 2DOS formulation because it is 4 order system of two equations. In two formulations, it is possible to separate symmetric and antisymmetric solutions in order to divise four the size of the computational domain. Finally, the discretized system is represented by a general eigenvalue problem $\boldsymbol{AX} = \omega \boldsymbol{X}$. The eigenvalues ω and its associated eigenvectors are determined by using the Arnoldi algorithm. In order to improve the precision of the computations and the convergence, different numerical techniques are used in particular in the numerical processing of the boundary conditions of 2DOS formulation.

6. Numerical results

If we want to compute the transient growth of a two-dimensional Poiseuille flow, it is necessary to know whole eigenvalues spectrum and not only the most unstable mode. In the two following sections, the spectrum of two-dimensional Poiseuille flow is computed by the numerical method descrived above.

Discretized Spectrum computation

A complete spectrum (superposition of the four modes) of the Poiseuille flow is shown in figure 1. The parameters are chosen according to Tatsumi and Yoshimura, 1990, so that the flow field is at its neutral point. The neutral mode is found to have a value of $\omega = (0.2112(2), 2.9(75).10^{-5})$, which is close to the result given by Theofilis, 1998, $\omega = (0.21167, 1.10^{-5})$ for a grid (60×40).

The spectrum in figure 2 is found to have a similar structure to that obtained with a one-dimensional approach. However important differences between the 1D and 2D spectrum are nevertheless observed. In the 1D spectrum, the eigenvalues are located on three main branches which have been labeled A, P and S by Mack

(1976). In the 2D spectrum these branches seem always exist but in a different way. The global Y shape characteristic to a 1D Poiseuille flow is roughly found in the 2D configuration. Although the P, S and A branches seem to be located at the same place as those in 1D, an important differences between the 1D and 2D spectrum are nevertheless observed. The P branch seems to be composed of two series of aligned modes, the S branch seems as complicated as the P branch with an additional difficulty because $\mid \omega \mid$ is not small and the convergence is weak. More number of point is necessary to reach the convergence. The structure of the A branch is more difficult to identify, this branch is strongly modified in comparison with that obtained in 1D approach. In the 1D configuration, the A branch is composed to several isolated modes. In 2D configuration, the A branch seem to be made up of aligned modes on segments of curve where one extremity of these segments corresponds to the locus of the 1D modes. These "segment of modes" are well converged. It is obvious in this figure that some part of the spectrum is not totally converged in spite of a significant number of points. In fact the P and S branches are not well converged. There are still plenty of disorganized modes lying between the left of the P branches and the right of the A branch and on the left of the S branch. These unconverged modes move if the grid was refined. Indeed, like it was say above the P branch of the spectrum seems to be composed of two series of line of modes. With a low discretisation, these lines of modes are truncated in bits of line, which "explodes" i.e. instead of forming a line of modes like in 1D case, this one diverge in two different directions like a fork. The length of the portion of lines increase when the number of points increase but this straightness is slow when the number of point increase. This phenomenon explains why there are many modes between the P and A branches organized more or less in lines. The same arguments apply to the S branch. In order to study the possible existence of two series of branches in the spectrum (for the P branch), a computation with a (65×51) points has been realized for $R_e = 10400$, $\beta = 0.91$ and $A = 2$. We note on figure 3 that P branch is in fact composed of two distinct P branches as we have supposed for $A = 5$ and these branches get closer when the A ratio increase and when $A \to \infty$ these branches collapse in unique branch. Moreover, in accordance with Tatsumi and Yoshimura, 1990 results, the basic flow is linearly stable when $A = 2$. Figure 3 shows this spectrum.

However, when $A = 5$ and for a grid (65×51) the spectrum is not sufficiently converged to be usable for the transient growth calculation in future. This figure establishes a very close relation with 1D and 2D instabilities. It seems like the transversal walls of the 2D duct only play a minor role on the spectrum structure at $A = 5$. Moreover, at this aspect ratio, the flow can not be considered to one-dimensional. Indeed, if we consider the critical Reynolds number it is equal to 10400 when $A = 5$ and it is only equal to 5772.22 when $A \to \infty$. The walls affect the flow by stabilizing it.

Comparison between 2DLNS and 2DOS

A comparison between the codes written with the Orr-Sommerfeld formulation (4) and the linearized Navier-Stokes equations (2) was performed. The spectra can be observed in figure 4. They are in very good correspondence. That validates each approache and their equivalence. In 1D approach, the LNS spectrum is equal to the sum of OS spectrum and Squire spectrum, in 2D approach, the 2DOS spectrum and 2DLNS spectrum are totally equivalent.

From a numerical point of view, the Orr-Sommerfeld formulation is much cheaper than the 2DLNS formulation, the memory space is divided by 3 and time by nearly 10. Convergence at a given number of points is equivalent in both cases. However, the Orr-Sommerfeld formulation is more sensible to numerical phenomenon, as discussed earlier. Moreover, the spectra shown here are obtained with derivative matrices containing information about parity. When the comparison is established for "natural" symmetric boundary conditions, it appears that the NSL spectrum is much converged than the Orr-Sommerfeld. The former looks very much like the one shown here whereas the latter is very polluted and the modes so scattered that the structure is barely recognizable. This is an other argument to say that the Orr-Sommerfeld formulation is very interesting in terms of costs, but also very sensible to numerical conditions.

7. Summary and conclusions

In this present paper, the complete spectrum of a two-dimensional Poiseuille flow has been shown. It was highlighted, as soon as $A \geq 5$, that the spectrum resulting from a one-dimensional approach and those resulting from a two-dimensional approach present strong similarities in their structures. Although, the existence of a finished aspect ratio A rather tends to stabilize the flow with respect to the monodimensional result ($A \to \infty$). The spectrum of Poiseuille flow seems to have three branches like its 1D counterpart. However, when $A < \infty$, the P branches seems to be double and tends towards a single branch when $A \to \infty$. This characteristic must undoubtedly play an important role in the mechanisms of transient growth of such flows because the modes resulting from the P and S branch in the one-dimensional case play a considerable part in the transient growth of the disturbances. However this study has shown, in spite of a powerful numerical method, that the convergence of the complete spectrum (in a rather broad field in ω) is not easily realizable, in particular in the vicinity of the P and S branches so that it is usable in a transient growth study. An improvement of the numerical methods must be carried out.

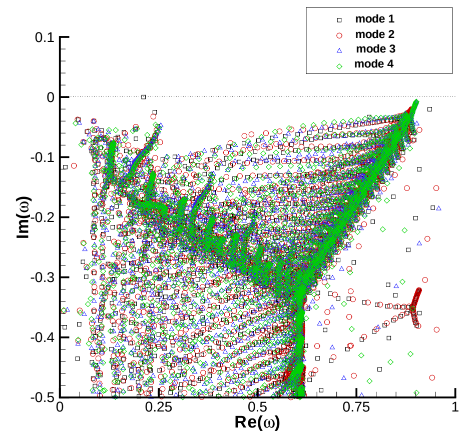

Figure 1. Complete spectrum of linear stability for a Poiseuille flow in a rectangular duct of aspect ratio $A = 5$. Grid (65×51), $R_e = 10400$ and $\beta = 0.91$.

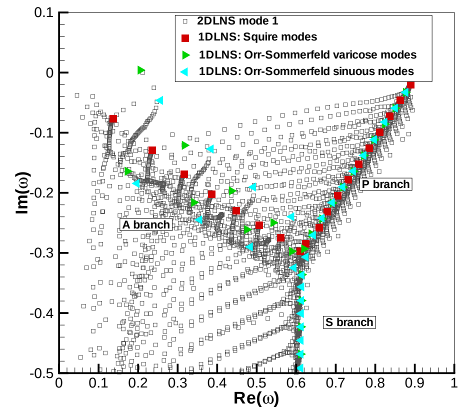

Figure 2. Poiseuille flow. Superposition of the linear stability spectra : ∘ 1D and □ 2D (mode 1). Aspect ratio $A = 5$. Grid (65×51) for 2D spectrum. $R_e = 10400$ and $\beta = 0.91$.

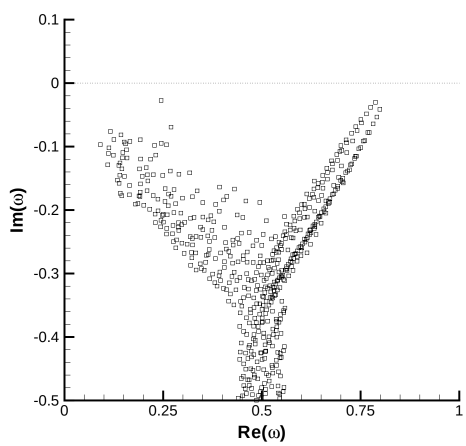

Figure 3. Discretized linear stability spectrum for aspect ratio $A = 2$, grid (65×51), $R_e = 10400$ and $\beta = 0.91$.

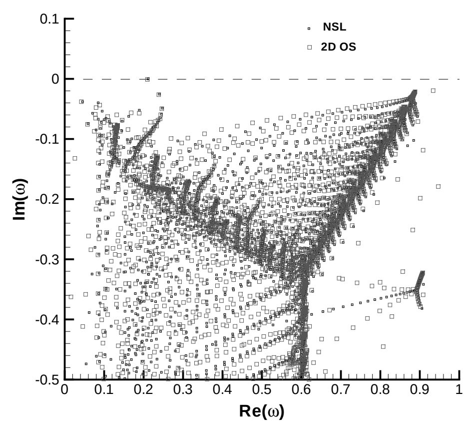

Figure 4. Poiseuille spectra (mode 1) obtained with different formulation of the equations of motion. □ : 2DOS; ∘ : 2DLNS. Grids (65×51). $A = 5$, $R_e = 10400$ and $\beta = 0.91$.

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