Model reduction using Balanced Proper Orthogonal Decomposition with frequential snapshots

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Abstract Many of the tools of flow control theory require model reduction to correctly capture the input-output behavior at stake. In this paper, we consider a model reduction based on balanced truncation using the method of snapshots. The particularity of this work is that these snapshots are computed in the frequency domain, allowing a reduction of their required number. This model is applied on the two-dimensional incompressible flow over a rounded backward facing step. We show that this model is able to catch the input-output behavior of the full system, a comparison with standard POD clearly demonstrates the superiority of this approach.

1 Introduction

Model reduction involves finding low-dimensional models that approximate the full high-dimensional dynamics. In terms of flow control, the systems are too large to apply optimal control tools so that a reduced-order model of the flow is necessary. Here we are interested in reduced order models based on Petrov-Galerkin projections. One can use global modes [1], proper orthogonal decomposition (POD) modes [2] or balanced modes [3] depending on the desired purpose. For control problems, the relevant quantity of interest is the input-output relation which can be optimally

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captured by balanced truncation [4]. This latter remains not yet computationally tractable for very large systems so that an approximated technique called balanced POD [5] has been introduced. It is based on a discretization of the controllability and observability gramians using snapshots. The present method relies on a frequential definition of the gramians and consequently frequential snapshots. BPOD and POD modes are computed in the case of a backward facing step flow to illustrate their ability to model an input-output relation.

2 Flow configuration

We consider the incompressible flow over a rounded backward facing step. The step consists of a circular part and is considered infinite in the spanwise direction. A uniform and unitary velocity field is prescribed at the inlet boundary and a laminar boundary layer starts developing on the lower boundary at $(x=-2, y=1)$. The upstream velocity and the step height are used to make all quantities non-dimensional. The Reynolds number is fixed to 600 so that the flow is globally stable to two-dimensional perturbations and no boundary layer instabilities are present. The stability analysis and simulations are performed using the same discretization (finite elements) which results in equations with approximately 400 000 degrees of freedom. The equations are linearized about a base flow (see figure 1) computed via a Newton method [7]. The input of the system is a vertical momentum impulse localized on a gaussian centered just in front of separation. The output is measured by the shear stress just after reattachment.

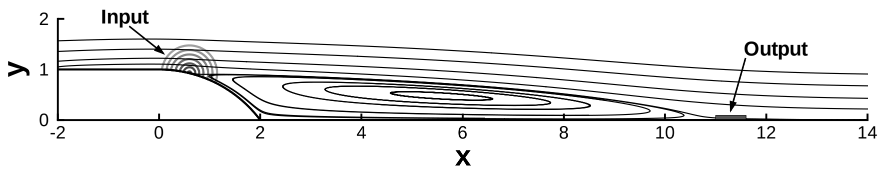


Fig. 1 Streamlines of the base flow at $Re = 600$ and localization of the input-output. This geometry comes from the experimental work of Duriez [6] realized at the PMMH laboratory.

3 Model reduction

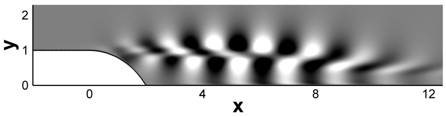
The dynamics of perturbations is given by the linear input-output system

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases} \tag{1}$$

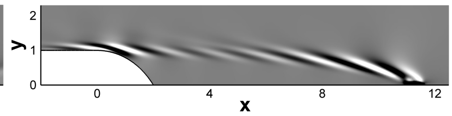
where $x(t)$ is the full state vector, A is the linearized Navier-Stokes operator, B is the control operator, C is the measure operator, $u(t)$ is the scalar input and $y(t)$ is the scalar output. The associated controllability and observability Gramians are then defined by

$$\begin{cases} G_c = \int_0^{+\infty} e^{At}BB^*e^{A^*t}dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty}(j\omega I - A)^{-1}BB^*(-j\omega I - A^*)^{-1}d\omega \\ G_o = \int_0^{+\infty} e^{A^*t}C^*Ce^{At}dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty}(-j\omega I - A^*)^{-1}C^*C(j\omega I - A)^{-1}d\omega \end{cases} \tag{2}$$

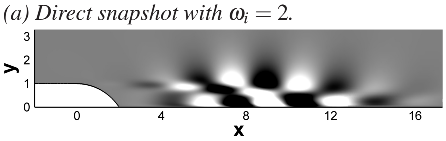
The discretization of the integrals is performed in the frequency domain namely on the r.h.s. of (2). Thus, the i^{th} direct snapshot is given by $(j\omega_i I - A)^{-1}B$ and its adjoint by $(-j\omega_i I - A^*)^{-1}C^*$. We have represented these snapshots on figure 2(a) and 2(b) for the pulsation $\omega_i = 2$. The BPOD modes are computed according to the technique introduced by [5] combining 279 equispaced snapshots from the direct and adjoint computations. The first BPOD modes are represented in figure 2(c) and 2(d). POD modes are computed in a similar manner (see [5]) using only the direct frequential snapshots.



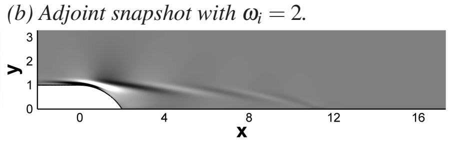
(a) Direct snapshot with $\omega_i = 2$.



(b) Adjoint snapshot with $\omega_i = 2$.



(c) First balanced mode.

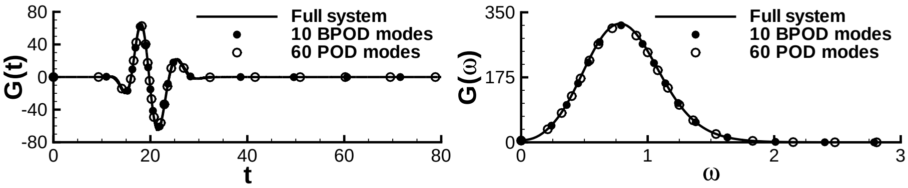


(d) First adjoint balanced mode.

Fig. 2 Snapshots and BPOD modes for the backward facing step flow.

4 Input-output response

The reduced models are computed from a Petrov-Galerkin projection of the system discretized equations. We have computed the inpulse response of the system as done by Barbagallo [8]. The temporal response is shown in figure 3(a) and its fourier transform in 3(b). The performance of the reduced order models to capture the input-output behavior is improved as the number of BPOD (or POD) modes increases. In both cases we succeeded in capturing the impulse response: the measurement signal of the reduced model matches accurately the measurement signal of the DNS. While only 10 BPOD modes are required, reduced models based on POD modes need at least 60 modes to achieve this goal.



(a) Temporal response $G(t)$. *(b) Frequential response $G(\omega)$.*

Fig. 3 Impulse responses for the different models versus two-dimensional direct numerical simulations. $G(t) = m(t)$ when $u(t) = \delta(t)$.

5 Conclusions and Outlook

In this paper, balanced POD has been performed on the flow over a backward facing step using snapshots in the frequency domain. With this technique, the controllability and observability gramians are accurately computed using fewer snapshots, reducing the computational time of the process. We showed that 10 BPOD modes (or 60 POD modes) are sufficient to catch the full input-output behavior of the system. These results show encouraging signs in terms of flow control (see [8]) and forecast models.

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